PG – 210

IV Semester M.Sc. Examination, June 2017
(CBCS)
MATHEMATICS
M401 T : Measure and Integration

Time : 3 Hours   Max. Marks : 70

Instructions : *i) Answer **any five (5)** full questions.*
*ii) **Each** question carries **equal** marks.*

1. a) Define a σ-algebra. Prove that every algebra on a set X is contained in the smallest σ-algebra on X.

   b) Prove that every open set E of real numbers is the union of a countable collection of disjoint open intervals. **(6+8)**

2. a) Define outer measure m* of a set. Prove that m* is countable subadditive and hence deduce that a countable set has measure zero.

   b) Construct Cantor ternary set from [0,1]. Show that Cantor ternary set is uncountable, but has measure zero. **(7+7)**

3. a) Define a measurable set. If $\mathscr{M}$ is the set of all measurable sets then prove that $\mathscr{M}$ is a σ-algebra.

   b) Let $\{E_i\}$ be an infinite increasing sequence of measurable sets then prove that

   $$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$$ **(7+7)**

4. a) Define a measurable function. Let $\{f_n\}$ be a sequence of measurable functions with the same domain of definition E. Then prove that the functions $\max\{f_1, f_2, ..., f_n\}$, $\min\{f_1, f_2, ..., f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\lim_n \sup f_n$, $\lim_n \inf f_n$ are all measurable functions.

   b) Prove that energy measurable function is almost a continuous function. **(7+7)**

5. a) Define Lebesgue integral of a function in comparison with Riemann integral. Is every Lebesgue integrable function Riemann integrable ? Justify your answer.

b) Define a simple function. Let f and g be bounded measurable functions defined on a set of finite measure. Then prove the following :

i) $\int_E af = a\int_E f$, for all a $\in \mathbb{R}$.

ii) $\int_E f + g = \int_E f + \int_E g$

iii) If f = g a.e. on E then $\int_E f = \int_E g$.

iv) If f $\leq$ g a.e. on E then $\int_E f \leq \int_E g$ and $\left|\int_E f\right| \leq \int_E |f|$.

v) If E_1 and E_2 are disjoint measurable sets of finite measure, then $\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$. **(6+8)**

6. a) If f is a non-negative measurable function and $\{E_i\}$ is a sequence of disjoint measurable sets with $\bigcup_{i=1}^{\infty} E_i = E$ then prove that $\int_E f dx = \sum_{i=1}^{\infty} \int_{E_i} f dx$.

b) State and prove monotone convergence theorem. Prove that monotone convergence theorem need not hold good for a decreasing sequence of functions. **(7+7)**

7. a) State and prove vital covering lemma.

b) Define absolute continuous function. If f(x) and g(x) are absolutely continuous functions, then prove that f $\pm$ g, f.g, and $\frac{f}{g}(g \neq 0)$ are absolutely continuous **(8+6)**

8. a) Establish the Holder's inequality. And deduce the Minkowski's inequality.

b) State and prove Riesz-Fischer throrem. **(7+7)**